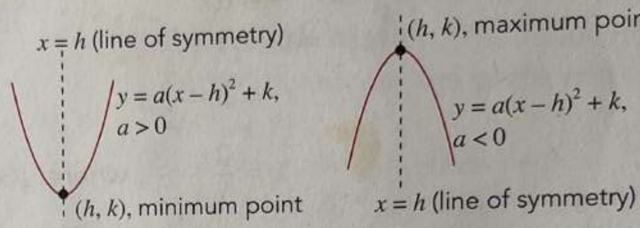
Summary

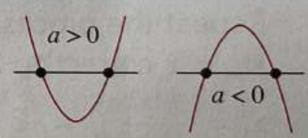
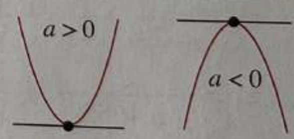
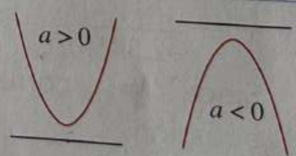
Quadratic Functions

Maximum and minimum values

Any quadratic function in general form $ax^2 + bx + c$ can be transformed to completed square form $a(x-h)^2 + k$.



Discriminant, roots and x-intercepts

Discriminant ($b^2 - 4ac$)	Nature of roots of $ax^2 + bx + c = 0$	Number of x-intercepts	Graph of $y = ax^2 + bx + c$
> 0	2 real and distinct roots	2	 The curve intersects the x-axis at 2 points.
$= 0$	2 real and equal (repeated) roots	1	 The curve intersects the x-axis at 1 point (or touches the x-axis).
< 0	No real roots	0	 The curve does not intersect the x-axis.

When discriminant < 0 and $a > 0 \Leftrightarrow y = ax^2 + bx + c > 0$ for all real values of x.
When discriminant < 0 and $a < 0 \Leftrightarrow y = ax^2 + bx + c < 0$ for all real values of x.

Summary

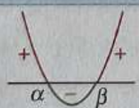
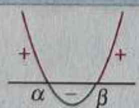
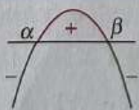
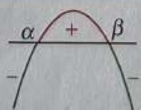
Equations and Inequalities

Simultaneous (linear and non-linear) equations
We can solve a linear equation and a non-linear equation simultaneously by substitution.

Intersection between a curve and a straight line
Finding the points of intersection between $y = ax^2 + bx + c$ and $y = mx + k$ is equivalent to solving the two equations simultaneously, which results in solving $ax^2 + bx + c = mx + k$, or $ax^2 + (b - m)x + (c - k) = 0$.

Discriminant of $ax^2 + (b - m)x + (c - k) = 0$	Nature of roots of $ax^2 + (b - m)x + (c - k) = 0$	Number of points of intersection between $y = ax^2 + bx + c$ and $y = mx + k$
> 0	2 real and distinct roots	2
$= 0$	2 real and equal (repeated) roots	1
$\geqslant 0$	2 real roots	1 or 2
< 0	No real roots	0

Quadratic inequalities
To solve a quadratic inequality in x, we could follow these steps.
Step 1: Bring all the terms to the left side of the inequality.
Step 2: Factorise the quadratic expression on the left side.
Step 3: Sketch the quadratic graph to deduce the range of values of x that satisfy the inequality.

Coefficient of x^2	Solve $ax^2 + bx + c > 0$	Solve $ax^2 + bx + c < 0$
$a > 0$	 The solution set is $x < \alpha$ or $x > \beta$.	 The solution set is $\alpha < x < \beta$.
$a < 0$	 The solution set is $\alpha < x < \beta$.	 The solution set is $x < \alpha$ or $x > \beta$.

Summary

Surds

Manipulation of surds

Using properties:

For $a, b > 0$, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\sqrt{a} \times \sqrt{a} = a$$

Rationalising the denominator:

If the denominator is in the form $\sqrt{k}$, multiply the numerator and denominator by $\sqrt{k}$.

If the denominator is in the form $a + b\sqrt{k}$, multiply the numerator and denominator by its conjugate $a - b\sqrt{k}$.

If the denominator is in the form $a\sqrt{h} + b\sqrt{k}$, multiply the numerator and denominator by its conjugate $a\sqrt{h} - b\sqrt{k}$.

Solving equations involving surds

We can square both sides of an equation involving surds to solve for the unknown, e.g.

$\sqrt{x-1} = 3 \Rightarrow \left(\sqrt{x-1}\right)^2 = 3^2$.

Always check the solutions by substituting the solutions into the original equation.

Finding unknowns in equations involving surds

We can use the following property of surds to find unknowns in an equation.

If $a + b\sqrt{k} = c + d\sqrt{k}$, where a, b, c, d are rational and $\sqrt{k}$ is irrational, then $a = c$ and $b = d$.

Journal Writing

Summary

Polynomials and Partial Fractions

Remainder Theorem

When f(x) is divided by $ax - b$, the remainder is $f\left(\frac{b}{a}\right)$. In particular, when f($x$) is divided by $x - a$, the remainder is f(a).

Factor Theorem

If $x - a$ is a factor of f(x), then f(a) = 0. Conversely if f(a) = 0, then $x - a$ is a factor of f(x).

Sum and difference of cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Partial fraction decomposition

f(x) and g(x) are polynomials, where g(x) ≠ 0.

If $\frac{f(x)}{g(x)}$ is proper, apply the rules in this table to perform partial fraction decomposition.

g(x) has	Write partial fraction(s) as
distinct linear factors: $ax + b$, $cx + d$	$\frac{A}{ax+b} + \frac{B}{cx+d}$
repeated linear factors: $ax + b$, $ax + b$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
irreducible quadratic factor: $x^2 + c^2$	$\frac{Ax+B}{x^2+c^2}$

If $\frac{f(x)}{g(x)}$ is improper, express it as the sum of a proper fraction and a polynomial first.

Factorising cubic polynomials

Follow these steps to factorise f(x).
Step 1: Use the Factor Theorem to find a linear factor of f(x), say $x - k$.
Step 2: Write f(x) as $(x - k)(ax^2 + bx + c)$.
Step 3: Substitute values of x or equate coefficients and/or constants to find the unknowns a, b and c.

Solving cubic equations

To solve f(x) = 0 for x, factorise f(x) and then equate each factor of f(x) to 0.

Summary

Exponential and Logarithmic Functions

Laws of indices

For $a, b > 0$, and rational numbers m and n,

$a^m \times a^n = a^{m+n}$
$\frac{a^m}{a^n} = a^{m-n}$
$(a^m)^n = a^{mn}$

} laws for the same base

$a^n \times b^n = (ab)^n$
$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

} laws for the same index

Solving exponential equations

- By equating indices:
  $a^x = a^n \Rightarrow x = n, a > 0$ and $a \neq 1$
- By substitution
- By taking logarithms

Definitions of indices

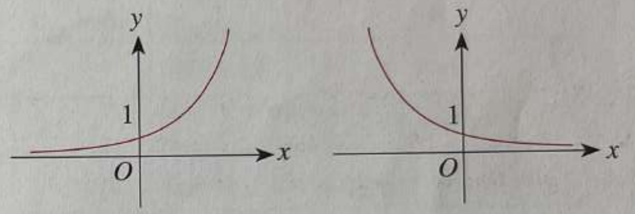
For $a > 0$, and positive integers p and q,

$a^0 = 1$	(zero index)
$a^{-p} = \frac{1}{a^p}$	(negative index)
$a^{\frac{1}{p}} = \sqrt[p]{a}$	(fractional index)
$a^{\frac{q}{p}} = \left(\sqrt[p]{a}\right)^q$	(fractional index)

Exponential graphs

Increasing y: $a > 1$

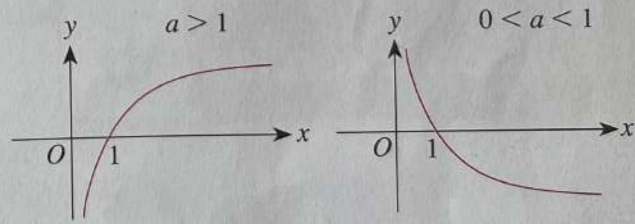
Decreasing y: $0 < a < 1$



Logarithmic graphs

Increasing y:

Decreasing y:



Laws of logarithms

If a, x, y are positive numbers and $a \neq 1$, then

$\log_a x^r = r \log_a x$ for any real number r (power law)

$\log_a (xy) = \log_a x + \log_a y$ (product law)

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ (quotient law)

If a, b, c are positive numbers and $a \neq 1$, $c \neq 1$, then

$\log_a b = \frac{\log_c b}{\log_c a}$ (change-of-base law)

Solving logarithmic equations

- By definition:
  $x = \log_a y \Leftrightarrow y = a^x$
- By using the property:
  $\log_a M = \log_a N \Rightarrow M = N$

Summary

Binomial Theorem

General Form

For a positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

General term, $T_{r+1} = \binom{n}{r}a^{n-r}b^r$, where $0 \leqslant r \leqslant n$

Binomial Coefficients in the Pascal's Triangle

$n = 0$: $\binom{0}{0}$

$n = 1$: $\binom{1}{0}$ $\binom{1}{1}$

$n = 2$: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$n = 3$: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$n = 4$: $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$n = 5$: $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

Special Form

$$(1+b)^n = \binom{n}{0}b^0 + \binom{n}{1}b^1 + \binom{n}{2}b^2 + \binom{n}{3}b^3 + \dots + \binom{n}{r}b^r + \dots + \binom{n}{n}b^n$$

General term, $T_{r+1} = \binom{n}{r}b^r$, where $0 \leqslant r \leqslant n$

Notations

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

Journal Writing

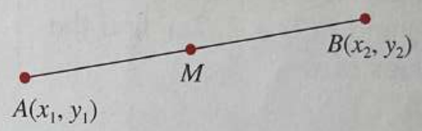
Summary

Parallel and perpendicular lines

- The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if and only if $m_1 = m_2$.
- The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if and only if $m_1m_2 = -1$ or $m_1 = -\frac{1}{m_2}$.

Coordinate Geometry

Midpoint of a line segment



The midpoint, M, of AB is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

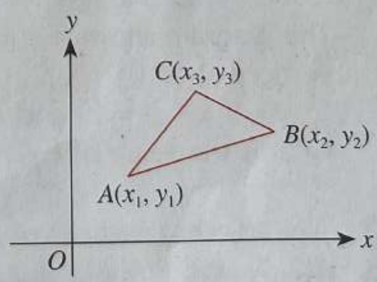
Areas of triangles and quadrilaterals

- Area of triangle ABC

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

where the points must be taken in an anticlockwise direction.

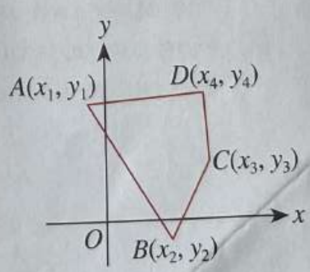


- Area of quadrilateral $ABCD$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4)$$

where the points must be taken in an anticlockwise direction.



Summary

Circles

Equation of a circle in standard form

$$(x - a)^2 + (y - b)^2 = r^2$$

Centre of circle = (a, b)
Radius of circle = r

Equation of a circle in general form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre of circle = $(-g, -f)$

Radius of circle = $\sqrt{f^2 + g^2 - c}$

To find the point of intersection between a line and a circle, solve equations of the circle and the line simultaneously.

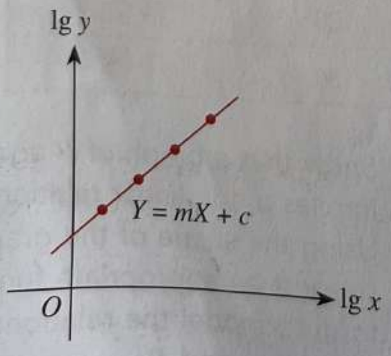
Summary

Applications of Straight Line Graphs

An equation of the form $y = ax^n$, where a and n are constants, can be expressed in linear form $Y = mX + c$.

$$y = ax^n$$
$$\lg y = n(\lg x) + \lg a$$

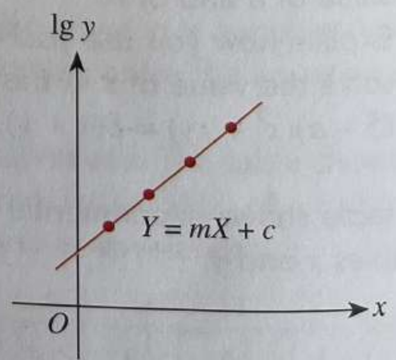
$Y = \lg y$, $X = \lg x$, $m = n$ and $c = \lg a$



An equation of the form $y = ab^x$, where a and b are constants, can be expressed in linear form $Y = mX + c$.

$$y = ab^x$$
$$\lg y = (\lg b)x + \lg a$$

$Y = \lg y$, $X = x$, $m = \lg b$ and $c = \lg a$



Summary

Trigonometric Functions

Special angles

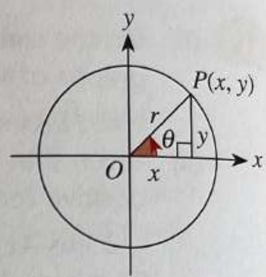
	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Trigonometric functions of any angle

$\sin\theta = \frac{y}{r}$ $\qquad$ $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, $\sin\theta \neq 0$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}$, $\cos\theta \neq 0$

$\tan\theta = \frac{y}{x}$, $x \neq 0$ $\qquad$ $\cot\theta = \frac{1}{\tan\theta}$, $\tan\theta \neq 0$



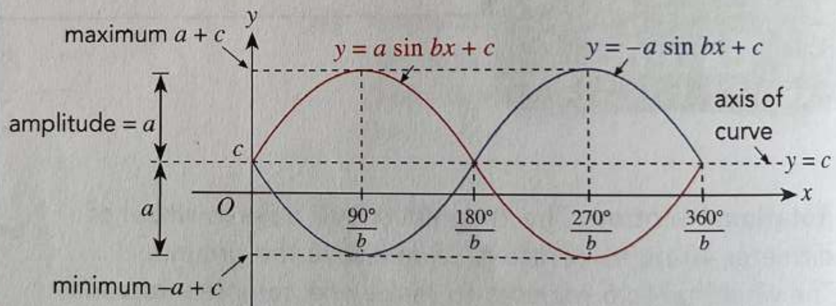
Negative angles

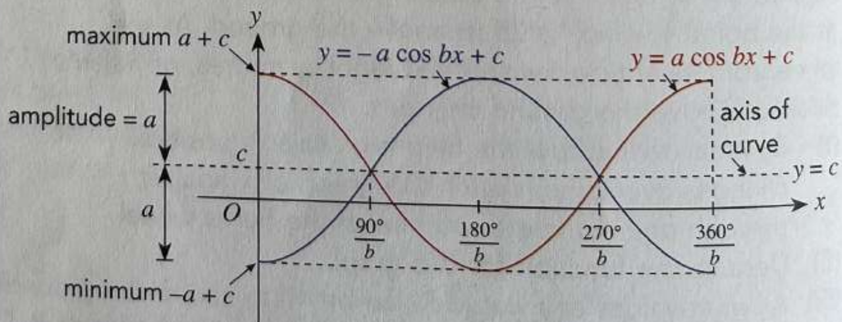
$\sin(-\theta) = -\sin\theta$

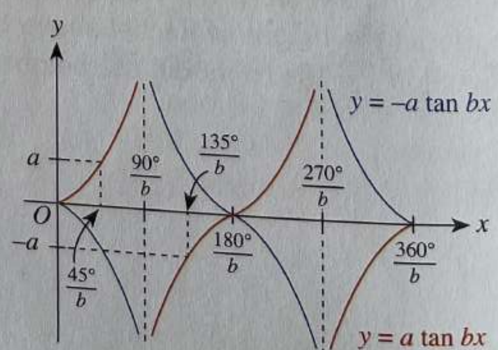
$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

Graphs of trigonometric functions







Principal values

- $-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$, where $-1 \leqslant x \leqslant 1$.
- $0 \leqslant \cos^{-1} x \leqslant \pi$, where $-1 \leqslant x \leqslant 1$.
- $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$, where x is any real number.

Trigonometric Identities and Equations

Simple identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, where $\sin\theta \neq 0$

Pythagorean identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

Addition Formulae

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Double Angle Formulae

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

R-Formulae

$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$

$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$

where $a > 0$, $b > 0$, α is acute,

$R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{b}{a}$.

Proving identities

Start with the side that looks more complex than the other side of the equation.

Apply trigonometric identities and algebraic manipulation.

Solving equations

Apply trigonometric identities.

Apply similar methods as solving algebraic equations, such as simplifying and factorising expressions.

Use the ASTC rule.

Trigonometric functions as models

$y = \pm a\sin bx + c$, where $a > 0$ and $b > 0$

$y = \pm a\cos bx + c$, where $a > 0$ and $b > 0$

Use the characteristics of sine and cosine functions to model real-world situations and periodic motion.