## Summary

# Differentiation

### The Power Rule

If $y = x^n$, where $n$ is a rational constant, then $\frac{dy}{dx} = nx^{n-1}$.

### The Constant Multiple Rule

If $f(x)$ is a function and $k$ is a constant, then $\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$.

### The Constant Rule

If $y = k$, where $k$ is a constant, then $\frac{dy}{dx} = 0$.

### The Chain Rule

If $y = f(u)$ and $u = g(x)$, and both $\frac{dy}{du}$ and $\frac{du}{dx}$ exist, then the derivative of the function $y = f[g(x)]$ exists and is given by $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

If $f(u) = u^n$ and $u = g(x)$, then $\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}$.

### The Sum Rule and Difference Rule

If $u$ and $v$ are functions of $x$, then $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$ and $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$.

### The Product Rule

If $u$ and $v$ are functions of $x$, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

### The Quotient Rule

If $u$ and $v$ are functions of $x$ and $v \neq 0$, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

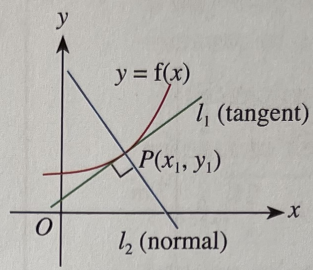
# Summary

## Tangents, Normals and Rates of Change

### Equations of tangents and normals

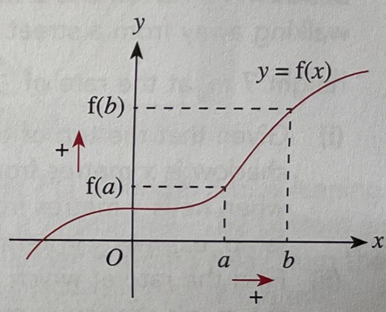
Equation of tangent at $P(x_1, y_1)$:
$y - y_1 = f'(x_1)(x - x_1)$

Equation of normal at $P(x_1, y_1)$:
$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$ if $f'(x_1) \neq 0$

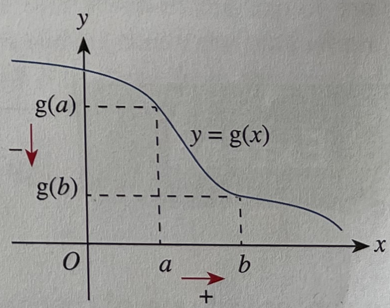



### Increasing and decreasing functions

If $\frac{dy}{dx} > 0$ for all $x$ in $a < x < b$, the function is increasing in $a < x < b$.




If $\frac{dy}{dx} < 0$ for all $x$ in $a < x < b$, the function is decreasing in $a < x < b$.




### Rates of change

The instantaneous rate of change of $y = f(x)$ with respect to $x$ at a point $(x_1, y_1)$ is the value of the derivative $\frac{dy}{dx}$ when $x = x_1$. The average rate of change of $y$ with respect to $x$ from $(x_1, y_1)$ to $(x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$.

### Connected rates of change

For a function $y = f(x)$ where variables $x$ and $y$ vary with time $t$,
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

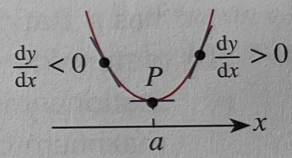
# Summary

**Maxima and Minima**

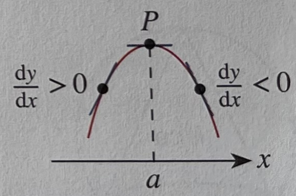
### First Derivative Test

Given a curve $y = f(x)$ and a stationary point $P$ at $x = a$,

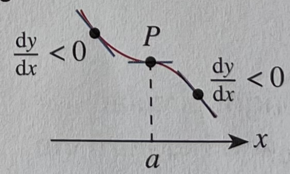
**(a)** if $\frac{dy}{dx}$ changes from negative to positive as $x$ increases through $a$, the point $P$ is a minimum point,



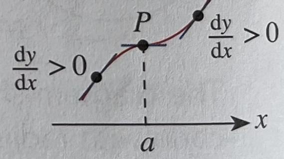

**(b)** if $\frac{dy}{dx}$ changes from positive to negative as $x$ increases through $a$, the point $P$ is a maximum point,




**(c)** if $\frac{dy}{dx}$ does not change sign as $x$ increases through $a$, the point $P$ is a stationary point of inflexion.




or




### Second Derivative Test

Given a curve $y = f(x)$ and a stationary point $P$ at $x = a$,

**(a)** if $\frac{d^2y}{dx^2} < 0$ at $x = a$, then $P$ is a maximum point,

**(b)** if $\frac{d^2y}{dx^2} > 0$ at $x = a$, then $P$ is a minimum point.

# Summary

**Differentiation of Trigonometric, Exponential and Logarithmic Functions**

### Derivatives of trigonometric functions

If $u$ is a function of $x$, $x$ is in radians, $a$, $b$ and $n$ are constants, and $n$ is rational, then

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$
- $\frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}$
- $\frac{d}{dx}(\tan u) = \sec^2 u \cdot \frac{du}{dx}$
- $\frac{d}{dx}[\sin (ax + b)] = a \cos (ax + b)$
- $\frac{d}{dx}[\cos (ax + b)] = -a \sin (ax + b)$
- $\frac{d}{dx}[\tan (ax + b)] = a \sec^2 (ax + b)$
- $\frac{d}{dx}(\sin^n x) = n \sin^{n-1} x \cdot \cos x$
- $\frac{d}{dx}(\cos^n x) = -n \cos^{n-1} x \cdot \sin x$
- $\frac{d}{dx}(\tan^n x) = n \tan^{n-1} x \cdot \sec^2 x$

### Derivatives of exponential functions

If $u$ is a function of $x$, and $a$ and $b$ are constants, then

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$
- $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$

### Derivatives of logarithmic functions

If $u$ is a function of $x$, and $a$ and $b$ are constants, then

- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$
- $\frac{d}{du}[\ln (ax + b)] = \frac{a}{ax + b}$

# Summary

## Integration

### Indefinite integrals

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
- $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
- $\int k f(x) \, dx = k \int f(x) \, dx$
- $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

where $a \neq 0$, $n$ is rational, $n \neq -1$ and $c$ is an arbitrary constant.

### Integration of trigonometric functions

- $\int \cos x \, dx = \sin x + c$
- $\int \cos (ax+b) \, dx = \frac{1}{a} \sin (ax+b) + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \sin (ax+b) \, dx = -\frac{1}{a} \cos (ax+b) + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \sec^2 (ax+b) \, dx = \frac{1}{a} \tan (ax+b) + c$

where $a \neq 0$ and $c$ is an arbitrary constant.

### Integration of exponential functions, $\frac{1}{x}$ and $\frac{1}{ax+b}$

- $\int e^x \, dx = e^x + c$
- $\int \frac{1}{x} \, dx = \ln x + c, \ x > 0$
- $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$
- $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln (ax+b) + c, \ ax+b > 0$

where $a \neq 0$ and $c$ is an arbitrary constant.

## Summary

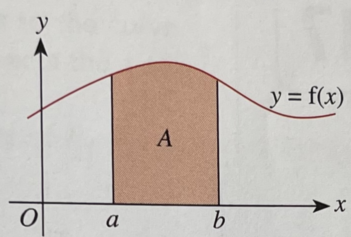
Applications of Integration

**Definite integrals**

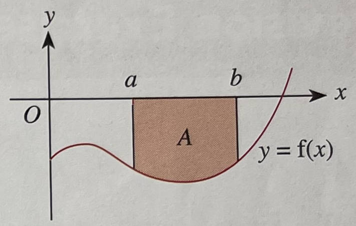
If $\frac{d}{dx}[F(x)] = f(x)$, then the definite integral of the function $f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

**Area between a curve and the *x*-axis**

In each diagram, the area $A$ bounded by a curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$ is as shown.




$A = \int_a^b f(x)\,dx$

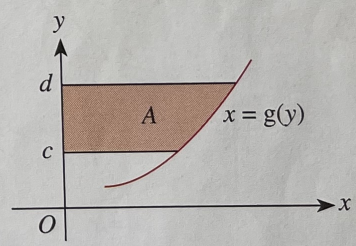



$A = -\int_a^b f(x)\,dx$

**Area between a curve and the *y*-axis**

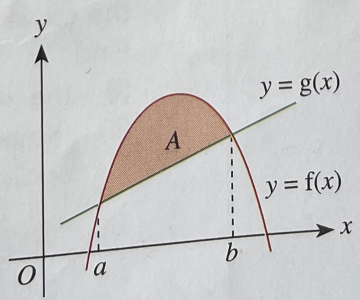
The area $A$ bounded by a curve $x = g(y)$, the $y$-axis and the lines $y = c$ and $y = d$ is given by

$A = \int_c^d x\,dy = \int_c^d g(y)\,dy.$

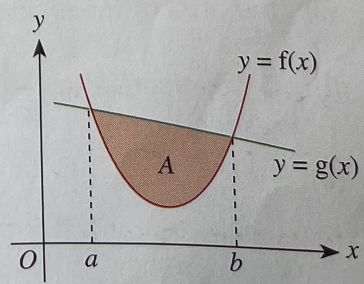



**Area between a curve and a line**

In each diagram, the area $A$ bounded by a curve $y = f(x)$ and a line $y = g(x)$ from $x = a$ to $x = b$ is as shown.




$A = \int_a^b [f(x) - g(x)]\,dx$




$A = \int_a^b [g(x) - f(x)]\,dx$

# Summary

## Kinematics

### Displacement

Displacement, $s$, of a moving particle at time $t$:

$$s = \int v \, \mathrm{d}t$$

### Velocity

Velocity, $v$, of a moving particle at time $t$:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad v = \int a \, \mathrm{d}t$$

### Acceleration

Acceleration, $a$, of a moving particle at time $t$:

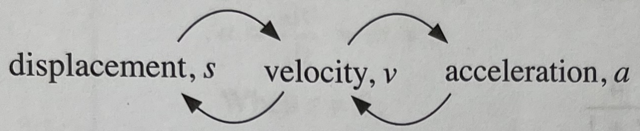
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

# Kinematics

REVISION notes

## Relationship between Displacement, Velocity and Acceleration

1. Differentiation: $\dfrac{ds}{dt}$ $\quad \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$




Integration: $s = \int v\,dt \qquad v = \int a\,dt$

2. Common Terms used in Kinematics:
   - Initial: $t = 0$
   - At rest: $v = 0$
   - Stationary: $v = 0$
   - Particle is at the fixed point: $s = 0$
   - Maximum/minimum displacement: $v = 0$
   - Maximum/minimum velocity: $a = 0$

3. Average speed = $\dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

4. To find the distance travelled in the first $n$ seconds:
   Step 1: Let $v = 0$ to find the value(s) of $t$.
   Step 2: Find $s$ for each of the values of $t$ found in Step 1.
   Step 3: Find $s$ for $t = 0$ and for $t = n$.
   Step 4: Draw the path of the particle on a displacement-time graph.

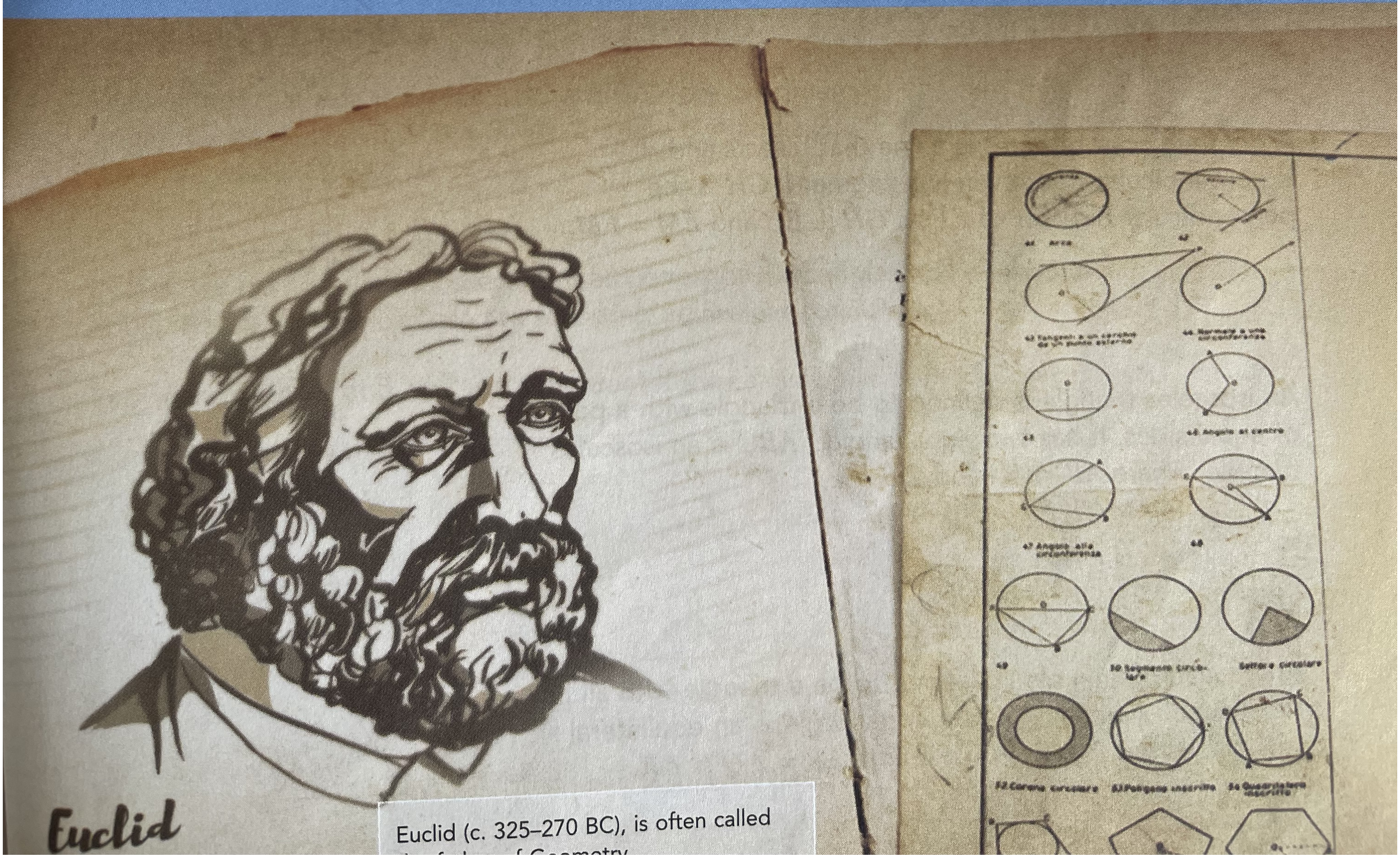
# CHAPTER 19

# Plane Geometry

We are already familiar with the idea of a geometrical proof, particularly in the topic on *Congruence* and *Similarity*. For example, we have learned to use tests such as SSS, ASA, SAS and RHS to show that two triangles are congruent (see table on page 163).

Ancient Greek mathematicians, such as Euclid (c. 325–270 BC), used logical reasoning to formulate geometrical proofs. He began with a collection of seemingly obvious geometrical statements, known as axioms, and used them as premises to prove other geometrical statements. After a statement is proven to be true, it is called a theorem. A theorem can in turn be used to prove other geometrical statements. The study of these geometrical axioms and their resulting theorems is called *Euclidean Geometry*.

In this chapter, we will explore further geometrical proofs using established theorems about properties of angles, triangles, quadrilaterals and circles. We will also learn two new theorems, the Midpoint Theorem and Alternative Segment Theorem.

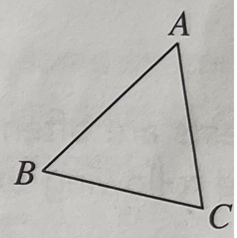



Euclid (c. 325–270 BC), is often called

joining an

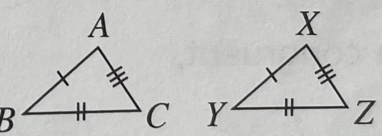
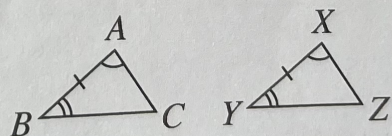
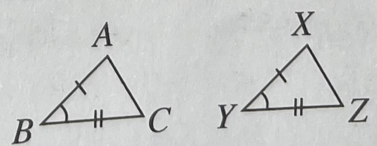
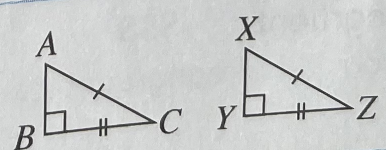
## Congruent Triangles

We learned that two triangles $ABC$ and $XYZ$ are congruent if angle $A$ = angle $X$, angle $B$ = angle $Y$, angle $C$ = angle $Z$, $AB = XY$, $BC = YZ$ and $AC = XZ$.



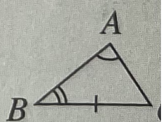

In fact, we do not need to check that all corresponding angles and correspo
equal. Recall that we can use the following equivalent conditions.

Two triangles $ABC$ and $XYZ$ are congruent if they satisfy one of the followin

| | | |
|---|---|---|
| Side – Side – Side<br>SSS | $AB = XY$<br>$BC = YZ$<br>$AC = XZ$ |  |
| Angle – Side – Angle<br>ASA | $\angle A = \angle X$<br>$AB = XY$<br>$\angle B = \angle Y$ |  |
| Side – Angle – Side<br>SAS | $AB = XY$<br>$\angle B = \angle Y$<br>$BC = YZ$ |  |
| Right angle – Hypotenuse – Side<br>RHS | $\angle B = \angle Y = 90°$<br>$AC = XZ$<br>$BC = YZ$ |  |

Think
Deep

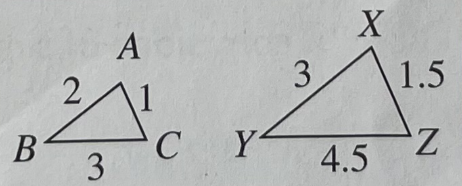
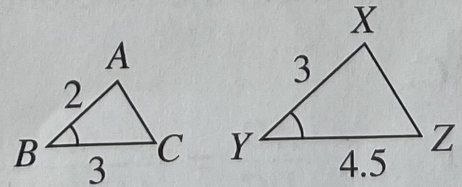
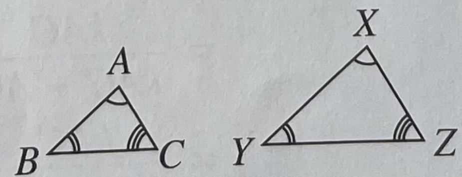
Given the f
of triangles
we can also
$\triangle ABC$ and
congruent?



## Similar Triangles

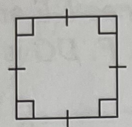
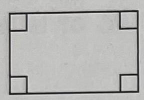
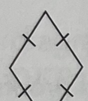
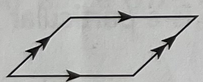
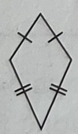
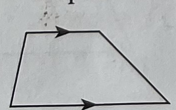
We have learned that two triangles are similar if they have the same shape but not necessarily the same size. To have the same shape means that all pairs of corresponding angles are equal and all pairs of corresponding sides are proportional.

Recall that two triangles *ABC* and *XYZ* are similar if they satisfy one of the following tests.

| | | |
|---|---|---|
| Side – Side – Side<br>SSS similarity | $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$ |  |
| Side – Angle – Side<br>SAS similarity | $\frac{AB}{XY} = \frac{BC}{YZ}$<br>$\angle B = \angle Y$ |  |
| Angle – Angle – Angle<br>AA (or AAA) similarity | $\angle A = \angle X$<br>$\angle B = \angle Y$<br>$\angle C = \angle Z$ |  |

If $\triangle ABC$ is congruent to $\triangle XYZ$, will $\triangle ABC$ also be similar to $\triangle XYZ$?

These similarity tests are often used to prove other geometrical statements.

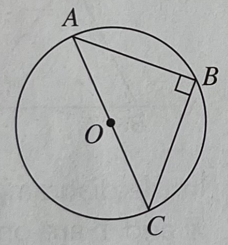
| Quadrilateral | Equivalent Conditions | Properties |
|---|---|---|
| Square  A quadrilateral with four equal sides and four right angles. | A quadrilateral is a square if it satisfies any one of these conditions.<br>• It has four equal sides and four right angles.<br>• Its diagonals are equal and bisect each other at right angles.<br>• It is a rectangle with one pair of equal adjacent sides.<br>• It is a rhombus with one right angle. | • It has parallel opposite sides.<br>• Each diagonal bisects a pair of opposite angles. |
| Rectangle  A quadrilateral with four right angles. | A quadrilateral is a rectangle if it satisfies any one of these conditions.<br>• It has four right angles.<br>• Its diagonals are equal and bisect each other.<br>• It is a parallelogram with one right angle. | • It has parallel opposite sides.<br>• It has equal opposite sides. |
| Rhombus  A quadrilateral with four equal sides. | A quadrilateral is a rhombus if it satisfies any one of these conditions.<br>• It has four equal sides.<br>• Its diagonals bisect each other at right angles.<br>• Each diagonal bisects a pair of opposite angles.<br>• It is a parallelogram with one pair of equal adjacent sides. | • It has parallel opposite sides.<br>• It has equal opposite angles. |
| Parallelogram  A quadrilateral with two pairs of parallel opposite sides. | A quadrilateral is a parallelogram if it satisfies any one of these conditions.<br>• It has two pairs of parallel opposite sides.<br>• It has two pairs of equal opposite sides.<br>• It has two pairs of equal opposite angles.<br>• It has one pair of parallel and equal opposite sides.<br>• Its diagonals bisect each other. | |
| Kite  A quadrilateral with two pairs of equal adjacent sides. | A quadrilateral is a kite if it satisfies any one of these conditions.<br>• It has two pairs of equal adjacent sides.<br>• One diagonal bisects the other diagonal at right angles.<br>• One diagonal bisects a pair of opposite angles. | • It has one pair of equal opposite angles. |
| Trapezium  A quadrilateral with one pair of parallel sides. | A quadrilateral is a trapezium if it has one pair of parallel sides. | |

# 19.3 Circle Theorems

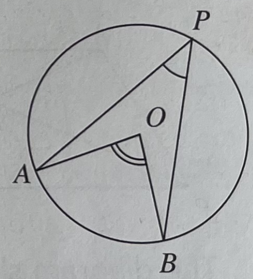
You Will Learn To
- Use angle properties of circles in geometrical proofs
- Use tangent properties of circles in geometrical proofs
- Apply the Alternate Segment Theorem

We have learned the following angle and tangent properties of circles.

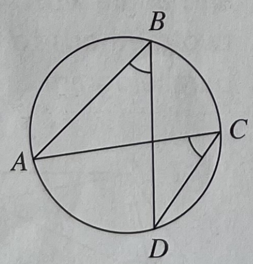
An angle in a semicircle is a right angle.
$\angle ABC = 90°$
($\angle$ in semicircle)



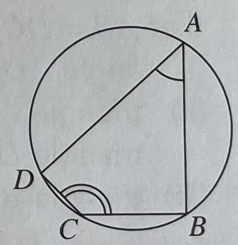

An angle at the centre is twice any angle at the circumference.
$\angle AOB = 2\angle APB$
($\angle$ at centre
$= 2\angle$ at circumference)



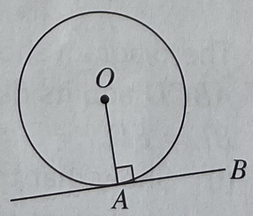

Angles in the same segment of a circle are equal.
$\angle ABD = \angle ACD$
($\angle$s in the same segment)



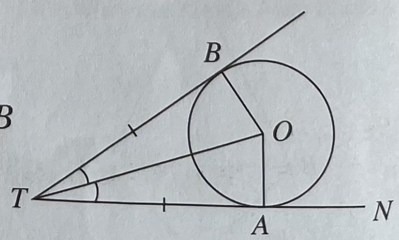

Angles in the opposite segments are supplementary.
$\angle BAD + \angle BCD = 180°$
($\angle$s in opposite segments)




A tangent of a circle is perpendicular to the radius drawn to the point of tangency.
$\angle OAB = 90°$
(tangent $\perp$ radius)




Given that $TA$ and $TB$ are tangents to the circle at $A$ and $B$ respectively, then $TA = TB$ and $\angle OTA = \angle OTB$.
(tangent properties)



# Summary

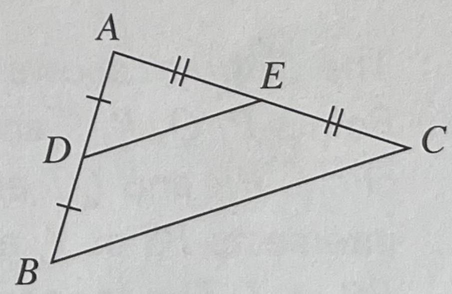
Plane Geometry

**Midpoint Theorem**

In triangle $ABC$, if $D$ and $E$ are midpoints of lines $AB$ and $AC$ respectively, then

**(a)** $DE \parallel BC$,

**(b)** $DE = \frac{1}{2}BC$.




**Alternate Segment Theorem**

If $AT$ is a tangent to the circle at $A$, then angle $BAT$ = angle $APB$.